

Enhancing an R-matrix.

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 27-11-1996

Abstract

In order to construct a representation of the tangle category one needs an enhanced R-matrix. In this paper we define a sufficient and necessary condition for enhancement that can be checked easily for any R-matrix. If the R-matrix can be enhanced, we also show how to construct the additional data that define the enhancement. As a direct consequence we find a sufficient condition for the construction of a knot invariant.

AMS Subject Classification: 16W30, 57M25.

Keywords & Phrases: Hopf algebra, quantum group, enhanced R-matrix, knot invariant, category of tangles.

1. Introduction

The title of this article contains two words that must be explained. The first one is the word *R-matrix*. An R-matrix is a $n^2 \times n^2$ matrix R that satisfies the Quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (1.1)$$

Here we have $R_{12} = R \otimes \text{id}$ and $R_{23} = \text{id} \otimes R$. By R_{13} we mean the following. Let V be a vector space of dimension n , with basis e_1, \dots, e_n . Then $R : V \otimes V \rightarrow V \otimes V$ is given by

$$R(e_i \otimes e_j) = R_{ij}^{kl} e_k \otimes e_l.$$

and $R_{13} : V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ by

$$R_{13}(e_i \otimes e_j \otimes e_k) = R_{ik}^{mn} e_m \otimes e_j \otimes e_n.$$

The complexity of equation (1.1) one better understands when one writes down the QYB-equation in terms of the matrix entries (R_{cd}^{ab}) . Equation (1.1) becomes

$$R_{k_1 k_2}^{ab} R_{u k_3}^{k_1 c} R_{v w}^{k_2 k_3} = R_{l_1 l_2}^{bc} R_{l_3 w}^{a l_2} R_{u v}^{l_3 l_1}. \quad (1.2)$$

The number of equations in (1.2) is n^6 , which makes a classification of all solutions of the QYB-equation very difficult, if not impossible. Nonetheless there are by now a great number of R-matrices known. The most famous ones come from the evaluation of the universal R-matrix of the universal enveloping algebras of the classical semi-simple complex Lie algebras in their fundamental representations. Inspired by the R-matrix associated to the Lie algebras $sl(n)$, Hazewinkel [4] classified all solutions of (1.2) under the restriction

$$R_{cd}^{ab} \neq 0 \implies \{a, b\} = \{c, d\}. \quad (1.3)$$

The question now arises what to do with these R-matrices. One thing one can do with them is try to construct knot and link invariants. Unfortunately we need some additional data for such a construction, which brings us to the other word in the title that must be explained.

Turaev [11] showed what extra data are needed for the construction of a knot invariant. He used the term *enhanced R-matrix*.

Definition 1.1. [11] *An enhanced R-matrix is a quadruple (S, μ, α, β) consisting of an invertible $n^2 \times n^2$ matrix S , a $n \times n$ matrix μ and two complex numbers $\alpha, \beta \in \mathbb{C}^*$ satisfying the following conditions:*

$$S_{12} S_{23} S_{12} = S_{23} S_{12} S_{23}, \quad (1.4)$$

$$S(\mu \otimes \mu) = (\mu \otimes \mu) S, \quad (1.5)$$

$$\text{Tr}_2(S^{\pm 1}(\mu \otimes \mu)) = \alpha^{\pm 1} \beta \mu. \quad (1.6)$$

The solutions of (1.4) have a simple relation with the solutions of (1.1). If R is a solution of (1.1), then both PR and RP are solutions of (1.4), where P is the permutation matrix $P_{cd}^{ab} = \delta_d^a \delta_c^b$. When written down in matrix entries we get

$$(PR)_{cd}^{ab} = R_{cd}^{ba}, \quad (RP)_{cd}^{ab} = R_{dc}^{ab}. \quad (1.7)$$

We say that an R-matrix R can be enhanced if there exists an enhanced R-matrix with $S = PR$ or $S = RP$. The symbol Tr_2 stands for the second trace, which in terms of matrix entries is defined by

$$\text{Tr}_2(A)_c^a = A_{cd}^{ad}. \quad (1.8)$$

This definition is independent of the basis with respect to which A is written (see [8]). When μ is invertible, then (1.6) is equivalent to

$$\text{Tr}_2(S^{\pm 1}(I \otimes \mu)) = \alpha^{\pm} \beta I. \quad (1.9)$$

If (S, μ, α, β) is an enhanced R-matrix, then $(\alpha^{-1}S, \beta^{-1}\mu, 1, 1)$ is one also. So both the factors α and β can be normalized to 1. Given such an enhanced R-matrix, Turaev constructed the link invariant

$$T_S(\xi) = \alpha^{-w(\xi)} \beta^{-m} \text{Tr}(\rho_S(\xi) \circ \mu^{\otimes m}). \quad (1.10)$$

Here ξ is a braid with m strands, $w(\xi) = \sum \varepsilon_i$ if $\xi = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_r}^{\varepsilon_r}$, where the σ_i are the standard generators of the braid group on m letters B_m and ρ_S is the braid representation in $(\mathbb{C}^n)^{\otimes m}$ defined by

$$\rho_S(\xi) = S_{i_1 i_1 + 1}^{\varepsilon_1} \cdots S_{i_r i_r + 1}^{\varepsilon_r}.$$

The invariant is well defined on links because the trace is actually a Markov trace, which means that its value is independent of the braid presentation of the link.

In [12] Turaev gives a slightly more restrictive definition of an enhanced R-matrix in order to get a sufficient condition for the construction of a representation of the category of tangles. There his definition is the following:

Definition 1.2. *Let V be a finite-dimensional vector space. An enhanced R-matrix is a pair (S, μ) where S is an automorphism of $V \otimes V$ and μ an automorphism of V satisfying conditions (1.4), (1.5) and (1.9) with $\alpha = \beta = 1$, and additionally*

$$(PS^{\mp 1})^{t_1}(I_{V^*} \otimes \mu)(S^{\pm 1}P)^{t_1}(I_{V^*} \otimes \mu^{-1}) = I_{V^* \otimes V}. \quad (1.11)$$

Here t_1 stands for the first transpose, which on matrices is defined by

$$(A^{t_1})_{cd}^{ab} = A_{ad}^{cb}.$$

With this definition he constructs a unique functor from the category of tangles to the category of vector spaces such that $F(+) = V$ and $F(-) = V^*$, for every enhanced R-matrix (S, μ) . When restricted to links Turaev's functor gives exactly (1.10). We postpone the definition of Turaev's functor to section 4, because it requires some of the basic results about the tangle category, which we explain in section 2.

The question that one asks naturally, after learning the meaning of the words enhanced R-matrix, is where this enhancement comes from. Turaev [11] enhanced the R-matrices associated to the classical semi-simple complex Lie algebras so that they match his definition, but he gave no explanation for his enhancement. Hazewinkel [4] gave a criterion for the enhancement of the R-matrices that satisfy the restriction (1.3) so that they match Turaev's definition, but he got his criterion in a purely combinatorial way that does not reveal the nature of enhancement. In this paper we explain enhancement in the language of category theory, which has become the most succesful way of looking at knot invariants coming from Lie algebras and their R-matrices, and give a simple criterion for whether an R-matrix can be enhanced or not. We actually show that for an R-matrix R that satisfies our criterion there exists a unique μ such that (PR, μ) and (RP, μ^{-1}) satisfy all the conditions (1.4), (1.5), (1.9) and (1.11). As a direct consequence our construction also gives an α and β such that (PR, μ, α, β) and $(RP, \mu^{-1}, \alpha, \beta)$ are enhanced R-matrices in the sense of definition 1.1. We would like to stress that our μ, α and β only depend on the R-matrix R , and not on any information coming from Lie algebras. However, in order to put this problem in terms of category theory, where we think it belongs, we have to assume that R is *biinvertible*. This means that not only R itself is invertible, but also its second transpose R^{t_2} , where

$$(R^{t_2})_{cd}^{ab} = R_{cb}^{ad}.$$

Although this seems to be a restriction, we will show in section 4 that any R-matrix that can be enhanced in the sense of def.1.2 is necessarily biinvertible. However, this is not true for R-matrices that can be enhanced in the sense of def.1.1. A counterexample is the permutation matrix P . It is easy to check that this matrix is not biinvertible and, since $P^2 = I$, we see that $(P^2, I, 1, n)$, with

n^2 the dimension of P , is an enhanced R-matrix in the sense of def.1.1. We don't know how many other known not biinvertible R-matrices can be enhanced and if there is a way to understand their enhancement. Despite this little gap in our explanation of enhancement, we still think our results are interesting enough. For example, in section 5 we show which 4×4 R-matrices ($n = 2$) can be enhanced and how. These R-matrices were classified by Hietarinto [6] (see also [9]). It would be interesting to see what kind of knot invariants they give. It would also be very interesting to apply our results to the R-matrices found by Cremmer and Gervais [1] and those classified by Van den Hijligenberg [7] and see if they can be enhanced and if so, what kind of knot invariants they give. The latter tried to generalize the work of Hazewinkel and classified a subclass of the R-matrices under the restriction

$$R_{cd}^{ab} \neq 0 \implies \{a, b\} = \{c, d\} \text{ or } a = \sigma(b), c = \sigma(d)$$

with $\sigma(i) = n + 1 - i$.

This paper is organized as follows. In section 2 we recall the basic facts about quasi-triangular Hopf algebras and braided categories. We followed [8] closely and refer for the details and the proofs to the same. In section 3 we explain the dual theory. This is the theory of dual quasi-triangular Hopf algebras and braided categories. For the details and the proofs we refer to [8] and [9]. Section 4 contains our own results. In section 5 we enhance all biinvertible 4×4 R-matrices ($n = 2$). Finally in the appendix we give an elementary proof of our main theorem for the reader who is interested in our results, but does not want to go through all the category theory in sections 2, 3 and 4.

2. The basic idea

Let H be a Hopf algebra with comultiplication Δ , counit ε and invertible antipode S . We say that H is quasi-triangular if there exists an invertible element $\mathcal{R} \in H \otimes H$ such that

$$\Delta^{\text{op}}(x) = \mathcal{R} \Delta(x) \mathcal{R}^{-1}, \quad (2.1)$$

$$(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (2.2)$$

$$(\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}. \quad (2.3)$$

Here Δ^{op} is the opposite comultiplication defined by $\Delta^{\text{op}} = \tau\Delta$, where τ is the flip $\tau(x \otimes y) = y \otimes x$. It is easy to check that these properties imply

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \quad (2.4)$$

$$(\varepsilon \otimes \text{id})(\mathcal{R}) = 1 = (\text{id} \otimes \varepsilon)(\mathcal{R}), \quad (2.5)$$

$$(S \otimes \text{id})(\mathcal{R}) = \mathcal{R}^{-1} = (\text{id} \otimes S^{-1})(\mathcal{R}), \quad (2.6)$$

$$(S \otimes S)(\mathcal{R}) = \mathcal{R}. \quad (2.7)$$

If such a \mathcal{R} exists, then (2.4) shows that every evaluation of \mathcal{R} in a representation of H satisfies the QYB-equation (1.1). That is why \mathcal{R} is called the universal R-matrix of H . Now suppose H is quasi-triangular with $\mathcal{R} = \sum r_i \otimes t_i$. Then there is a well known lemma that says that S^2 is an inner automorphism of H .

Lemma 2.1. *Under the previous hypothesis, the elements $u = \sum S(t_i)r_i$ and $v = S(u)$ are invertible elements in H such that*

$$S^2(x) = uxu^{-1} = v^{-1}xv. \quad (2.8)$$

The element $uv = vu$ is central in H , and satisfies

$$\Delta(uv) = (\mathcal{R}_{21}\mathcal{R})^{-2}(uv \otimes uv) \quad (2.9)$$

In order to see the connection with knot invariants we have to consider knots as morphisms in a special tensor category \mathcal{T} , the category of tangles. The objects of \mathcal{T} are finite sequences of $+$ and $-$ signs. Their tensor product is the sequence that we get by putting one sequence after the other. The identity object is just the empty sequence. If we put two such sequences one above the other, then a morphism in \mathcal{T} is an equivalence class of oriented tangles between them. By this we mean that, taking an arbitrary tangle in the equivalence class, all $+$ and $-$ signs are either the head or the tail of a strand of this tangle. The head of a strand is attached to a $+$ sign if this head is pointing downward and it is attached to a $-$ sign if the head is pointing upward. A tail of a strand is attached to a $+$ sign if it is pointing downward and attached to a $-$ sign if it is pointing upward. Two tangles are equivalent if one can be obtained from the other by only applying homotopies of the tangle diagram and the Reidemeister moves 1,2 and 3. In fig.2.1. we show an example.

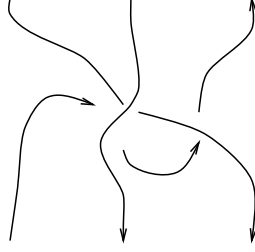


Figure 2.1: A tangle.

The composition of two tangles is obtained by putting the first tangle on top of the second. This means that we 'read' the tangles from bottom to top. The tensor product of two tangles is given by juxtaposition and the identity endomorphism of an object is just the set of straight vertical strands with orientation determined by the signs in the sequence. It is an easy exercise to show that these objects and morphisms define a tensor category, with the empty set being the identity object. In fact they have more structure, which makes tangles intimately related to the representation theory of quasi-triangular Hopf algebras. We can define something called a *braiding* in the category of tangles.

Definition 2.2. A braiding in a tensor category C is a family of natural isomorphisms $c_{V,W} : V \otimes W \rightarrow W \otimes V$, indexed by all pairs of objects $V, W \in \text{Obj}(C)$, such that

$$c_{U,V \otimes W} = (\text{id}_V \otimes c_{U,W}) (c_{U,V} \otimes \text{id}_W) \quad (2.10)$$

$$c_{U \otimes V, W} = (c_{U,W} \otimes \text{id}_V) (\text{id}_U \otimes c_{V,W}) \quad (2.11)$$

The naturality of c means that

$$c(f \otimes g) = (g \otimes f) c,$$

where f and g are morphisms in C and c should be understood with the right subscripts. It is not difficult to see that (2.10) and (2.11) imply the following identity

$$(c_{V,W} \otimes \text{id}_U) (\text{id}_V \otimes c_{U,W}) (c_{U,V} \otimes \text{id}_W) = (\text{id}_W \otimes c_{U,V}) (c_{U,W} \otimes \text{id}_V) (\text{id}_U \otimes c_{V,W}). \quad (2.12)$$

If we take $U = V = W$, then identity 2.12 is similar to identity 1.4. That is why such a c is sometimes called a Yang-baxter operator on V . The braiding in the category of tangles is defined in fig.2.2. The inverse of c one gets by changing the overcrossings in undercrossings.

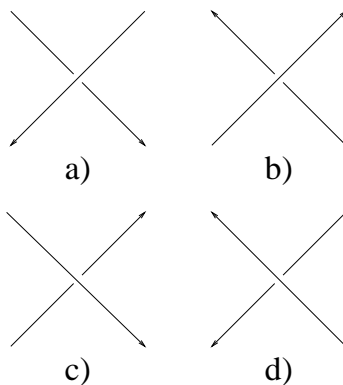


Figure 2.2: a) $c_{+,+}$ b) $c_{-,-}$ c) $c_{-,+}$ d) $c_{+,-}$.

In the same way as certain algebras or groups can be presented by a finite set of generators and relations between them there are categories that can be presented by a finite set of generating morphisms and relations between these morphisms. The category \mathcal{T} can be presented by the generators $X^+, X^-, \cup, \cup^-, \cap, \cap^-$ (see fig.2.3). For the defining relations see [12] or [8].

Another *braided category* is the category of finite dimensional H -modules, where H is a quasi-triangular Hopf algebra (Kassel calls them *braided* Hopf algebras, but the same name is used by Majid [9] for Hopf algebras in a braided category, which is why we prefer to use the 'old' name). The theorem is actually a bit stronger than that.

Theorem 2.3. *Let H be a Hopf algebra and ${}_H\mathcal{M}$ its category of finite dimensional left modules. If H is quasi-triangular, then ${}_H\mathcal{M}$ is a braided category. Conversely, if ${}_H\mathcal{M}$ is a braided category and H is finite dimensional, then H is a quasi-triangular Hopf algebra.*

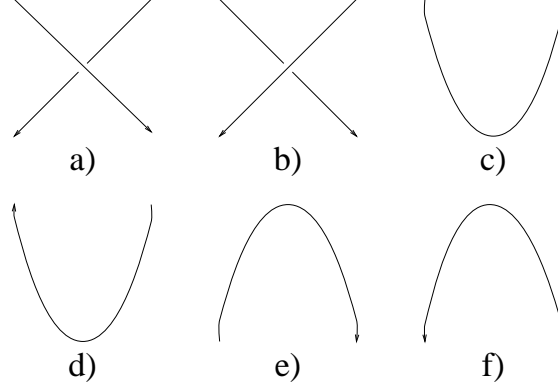


Figure 2.3: a) X^+ b) X^- c) \cup d) \cup^- e) \cap f) \cap^- .

Sketch of a Proof. Suppose H is quasi-triangular and \mathcal{R} is the universal R-matrix of H , then we define the braiding c in ${}_H\mathcal{M}$ by

$$c_{V,W}(v \otimes w) = \tau_{V,W}(\mathcal{R}(v \otimes w)), \quad (2.13)$$

where $\tau_{V,W}: V \otimes W \rightarrow W \otimes V$ is the flip operator. It is easy to check that this really defines a braiding. Conversely, suppose that ${}_H\mathcal{M}$ is braided and H is finite dimensional. Then H is a finite dimensional H -module with the action defined by its multiplication. We define the invertible element

$$\mathcal{R} = \tau_{H,H}(c_{H,H}(1 \otimes 1)).$$

It is easy to check that H becomes quasi-triangular with universal R-matrix \mathcal{R} ■

Instead of the category of left H -modules, we could also consider the category of right H -modules \mathcal{M}_H . The theorem above then goes through as in the case of left modules, but, of course, we now have to define

$$c_{V,W}(v \otimes w) = \tau_{V,W}((v \otimes w) \mathcal{R}).$$

If we consider framed tangles instead of normal tangles we get the category of framed tangles \mathcal{FT} . Its objects are the same as those of \mathcal{T} , but the morphisms

are equivalence classes of framed tangles. The framing of a tangle is an equivalence class of normal vector fields on the tangle. These framed tangles can be depicted as normal tangles with the convention that the framing comes orthogonally out of the paper. With respect to the orientation nothing changes but the equivalence relation has to be modified. Two framed tangles are said to be equivalent if one can be obtained from the other by isotopy of their diagrams, Reidemeister moves 2 and 3, and the modified Reidemeister move 1' depicted in fig.2.4.

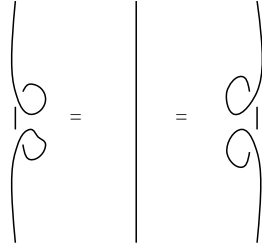


Figure 2.4: Reidemeister move 1'.

This category is a tensor category in the same way as \mathcal{T} and the braiding in \mathcal{T} defines a braiding in \mathcal{FT} as well. It is also possible to define a last bit of extra structure on our category, called *left duality* and *twist*.

Definition 2.4. A tensor category C with identity object I is a category with left duality if for every object V in C there exists an object $*V$ and morphisms

$$b_V : I \rightarrow V \otimes *V \text{ and } d_V : *V \otimes V \rightarrow I$$

in the category C such that

$$(\text{id}_V \otimes d_V)(b_V \otimes \text{id}_V) = \text{id}_V \text{ and } (d_V \otimes \text{id}_{*V})(\text{id}_{*V} \otimes b_V) = \text{id}_{*V}. \quad (2.14)$$

Using this definition we can define the *transpose* $*f : *V \rightarrow *U$ of a morphism $f : U \rightarrow V$ in C by

$$*f = (d_V \otimes \text{id}_{*U})(\text{id}_{*V} \otimes f \otimes \text{id}_{*U})(\text{id}_{*V} \otimes b_U).$$

In this way left duality defines a functor $*$: $C \rightarrow C$. In general this functor is not an involution, which is why not every R-matrix gives a representation of the category of tangles. We will explain this in section 4.

Theorem 2.5. *In a braided category with left duality there are natural equivalences $u, v^{-1} \in \text{Nat}(\text{Id}, *^2)$ defined by*

$$\begin{aligned} u_V &= (d_V \otimes \text{id}) (c_{V,*V} \otimes \text{id}) (\text{id} \otimes b_{*V}), \\ u_V^{-1} &= (\text{id} \otimes d_{*V}) (c_{**V,V} \otimes \text{id}) (\text{id} \otimes b_V), \\ v_V &= (d_{*V} \otimes \text{id}) (\text{id} \otimes c_{V,*V}) (\text{id} \otimes b_V), \\ v_V^{-1} &= (d_V \otimes \text{id}) (\text{id} \otimes c_{**V,V}) (b_{*V} \otimes \text{id}), \end{aligned}$$

obeying

$$\begin{aligned} u_{V \otimes W} &= c_{V,W}^{-1} c_{W,V}^{-1} (u_V \otimes u_W), \\ v_{V \otimes W} &= c_{V,W}^{-1} c_{W,V}^{-1} (v_V \otimes v_W), \end{aligned}$$

and such that

$$**f = u_W \circ f \circ u_V^{-1} = v_W^{-1} \circ f \circ v_V, \quad \forall f : V \rightarrow W.$$

In general left duality is only unique up to an isomorphism. If $(\times, b^\times, d^\times)$ defines another left duality, then one can define an isomorphism $f_V :^\times V \rightarrow *V$ for every object V by

$$f_V = (d_V^\times \otimes \text{id}) (\text{id} \otimes b_V), \quad f_V^{-1} = (d_V \otimes \text{id}) (\text{id} \otimes b_V^\times).$$

So we get

$$d_V^\times = d_V (f_V \otimes \text{id}), \quad b_V^\times = \left(\text{id} \otimes f_V^{-1} \right) b_V.$$

There is a similar notion of right duality.

Definition 2.6. *A tensor category C with identity object I is a category with right duality if for every object V in C there exists an object V^* and morphisms*

$$b'_V : I \rightarrow V^* \otimes V \text{ and } d'_V : V \otimes V^* \rightarrow I$$

in the category C such that

$$\left(\text{id}_{V^*} \otimes d'_V \right) \left(b'_V \otimes \text{id}_{V^*} \right) = \text{id}_{V^*} \text{ and } \left(d'_V \otimes \text{id}_V \right) \left(\text{id}_V \otimes b'_V \right) = \text{id}_V. \quad (2.15)$$

Of course right duality is also unique up to isomorphism. If H is a Hopf algebra with invertible antipode, then we can take the left dual ${}^*V = \text{Hom}(V, \mathbb{C})$, the dual vector space, of every finite dimensional H -module V and make it into a left H -module by defining

$$x \triangleright f(y) = f(S(x)y)$$

for every $f \in {}^*V$, every $x \in H$ and every $y \in V$. The maps d_V and b_V are simply the evaluation and coevaluation map respectively. If H is quasi-triangular, we find the isomorphisms u_V and v_V to be the actions of $u \in H$ and $v \in H$ on V .

The next notion we want to introduce is that of a *twist*.

Definition 2.7. A twist in a braided tensor category C with left duality is a family $\theta_V : V \rightarrow V$ of natural isomorphisms indexed by all objects V in C such that

$$\theta_{V \otimes W} = (\theta_V \otimes \theta_W) c_{W,V} c_{V,W}, \quad (2.16)$$

$$\theta_{*V} = {}^*(\theta_V) \quad (2.17)$$

for all objects V, W in C .

A braided tensor category with left duality and twist is called a *ribbon category*. In a ribbon category we also have right duality. We just take $V^* = {}^*V$ and define

$$b'_V = (\text{id}_{V^*} \otimes \theta_V) c_{V,V^*} b_V, \quad (2.18)$$

$$d'_V = d_V c_{V,V^*} (\theta_V \otimes \text{id}_{V^*}). \quad (2.19)$$

Before we give two examples of ribbon categories, we first state a technical lemma that we will need in section 4.

Lemma 2.8. For any object V in a ribbon category, we have

$$\begin{aligned} \theta_V^{-2} &= (d_V \otimes \text{id}_V) (\text{id}_{*V} \otimes c_{V,V}^{-1}) (c_{V,*V} b_V \otimes \text{id}_V) \\ &= (d_V c_{V,*V} \otimes \text{id}_V) (\text{id}_V \otimes c_{V,*V} b_V) \\ &= (\text{id}_V \otimes d_V c_{V,*V}) (c_{V,*V}^{-1} \otimes \text{id}_{*V}) (\text{id}_V \otimes b_V) \end{aligned} \quad (2.20)$$

Of course our examples of ribbon categories are again the category of framed tangles and the category of H -modules. We first define the duality and the twist in \mathcal{FT} . (Framed tangles are often called ribbons, so that is where the name *ribbon category* comes from.) Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ be a finite sequence of $+$ and $-$ signs.

Define the dual object ${}^*\varepsilon = (-\varepsilon_n, \dots, -\varepsilon_1)$. The morphisms $b_\varepsilon : \emptyset \rightarrow \varepsilon \otimes {}^*\varepsilon$ and $d_\varepsilon : {}^*\varepsilon \otimes \varepsilon \rightarrow \emptyset$ are the framed tangles depicted in fig.2.5, their orientation being completely determined by the signs in ε . The relations (2.14) are easy to check in this case. Note that the transpose *L of a tangle L is obtained by rotation of the whole diagram through an angle π .

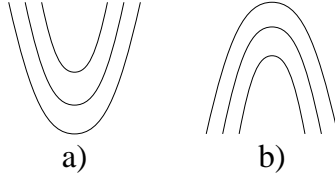


Figure 2.5: a) b_ε b) d_ε .

The twist on $+$, denoted by φ , we define in fig.2.5. The twist on an arbitrary object is then defined by the formulas (2.16) and (2.17). The category \mathcal{FT} can be presented by $X^+, X^-, \cup, \cup^-, \cap, \cap^-$ and all the same defining relations as in \mathcal{T} except one: the one that corresponds to Reidemeister move 1. This one has to be replaced by a relation that expresses Reidemeister move 1'. The exact definition of this relation in \mathcal{FT} we leave to the reader.



Figure 2.6: φ_+ .

Now we can formulate a very important property of \mathcal{FT} , called the *universality*

property

Theorem 2.9. *Let C be a ribbon category and V an object in C . Then there exists a unique functor $F_V : \mathcal{FT} \rightarrow C$ preserving the braiding, the left duality and the twist, such that $F_V(+)=V$ and $F_V(-)=*V$.*

The functor F_V has the following properties:

$$F(X^+) = c_{V,V}, \quad F(\varphi) = \theta_V, \quad F(\cup) = b_V, \quad F(\cap) = d_V;$$

$$F(X^-) = c_{V,V}^{-1}, \quad F(T^+) = c_{V,*V}^{-1}, \quad F(T^-) = c_{*V,V},$$

$$F(Y^+) = c_{*V,V}^{-1}, \quad F(Y^-) = c_{V,*V},$$

$$F(Z^+) = c_{*V,*V}, \quad F(Z^-) = c_{*V,*V}^{-1}, \quad F(\varphi^-) = \theta_V^{-1}.$$

The tangles Z^+, Y^+, T^+ are depicted in fig.2.6. Their inverses one gets by changing overcrossings in undercrossings.

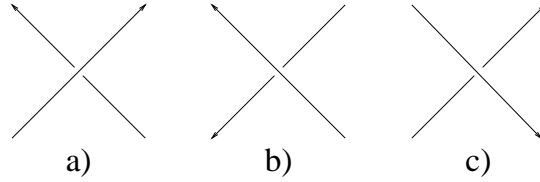


Figure 2.7: a) Z^+ b) Y^+ c) T^+ .

The values on \cup^- and \cap^- can be easily computed from the formulas

$$\cup^- = (\uparrow \otimes \varphi^-) \circ Y^+ \circ \cup,$$

$$\cap^- = \cap \circ Y^- \circ (\varphi \otimes \uparrow).$$

From (2.18) and (2.19) we get $F(\cup^-) = b'_V$ and $F(\cap^-) = d'_V$. It follows that $X^\pm, Y^\pm, \varphi^\pm, \cup, \cap$ is another set of generators for \mathcal{FT} . Conversely, we can also express Z^\pm, Y^\pm, T^\pm in terms of $X^\pm, \cup^\pm, \cap^\pm$.

Lemma 2.10. *The following relations hold in the categories \mathcal{T} and \mathcal{FT} :*

$$\begin{aligned} Y^\pm &= (\uparrow\downarrow \cap^-) (\uparrow X^\pm \uparrow) (\cup^- \downarrow\uparrow), \\ T^\pm &= (\cap \downarrow\uparrow) (\uparrow X^\pm \uparrow) (\uparrow\downarrow \cup), \\ Z^\pm &= (\cap \uparrow\uparrow) (\uparrow \cap \downarrow\uparrow\uparrow) (\uparrow\uparrow X^\pm \uparrow\uparrow) (\uparrow\uparrow\downarrow \cup \uparrow) (\uparrow\uparrow \cup), \\ Z^\pm &= (\uparrow\uparrow \cap^-) (\uparrow\uparrow\downarrow \cap^- \uparrow) (\uparrow\uparrow X^\pm \uparrow\uparrow) (\uparrow \cup^- \downarrow\uparrow\uparrow) (\cup^- \uparrow\uparrow). \end{aligned}$$

Now suppose that we have a ribbon category C and a specified object V in C . Then we get an invariant of framed links with values in $\text{End}(I)$, with I the identity object in C , if we consider framed links as morphisms in \mathcal{FT} from the empty set to itself. If C is a subcategory of the category $\mathcal{V}_{\mathbb{C}}$, the category of complex vector spaces, we get an invariant with values in \mathbb{C} . An important question is now whether or not we can find such ribbon categories, because all the machinery above only gives us a concrete framed knot invariant if we come up with a concrete ribbon category different from \mathcal{FT} . Before we show an example we first give one more definition.

Let H be a braided Hopf algebra with invertible antipode S . We defined in lemma 2.1 the central elements u, v such that $S^2(x) = uxu^{-1} = v^{-1}xv$.

Definition 2.11. *A quasi-triangular Hopf algebra H is called a ribbon algebra if there exists an invertible central element $\theta \in H$, that we call the ribbon element, such that*

$$\theta^2 = vu, \quad \Delta(\theta) = (\mathcal{R}_{21}\mathcal{R})^{-1}(\theta \otimes \theta), \quad \varepsilon(\theta) = 1, \quad S(\theta) = \theta. \quad (2.21)$$

This definition is exactly the one needed to make the following theorem hold.

Theorem 2.12. *For any ribbon algebra H , the tensor category ${}_H\mathcal{M}$ is a ribbon category with twist θ_V given on any H -module V by the multiplication by the inverse of the ribbon element.*

Conversely, if H is a finite-dimensional braided Hopf algebra and the braided category ${}_H\mathcal{M}$ is a ribbon category, then H is a ribbon algebra with ribbon element defined by $\theta = \theta_H(1)^{-1}$.

Of course there is a similar theorem for \mathcal{M}_H . Note that a priori there may be more than one ribbon element in the same quasi-triangular Hopf algebra.

To give a first example of a ribbon algebra we consider Sweedler's four dimensional Hopf algebra \mathcal{S} . This Hopf algebra is generated by

$$1, x, y$$

with defining relations

$$x^2 = 1, y^2 = 0, xy + yx = 0.$$

The Hopf algebra structure is given by

$$\Delta(x) = x \otimes x, \Delta(y) = 1 \otimes y + y \otimes 1, \varepsilon(x) = 1, \varepsilon(y) = 0$$

and

$$S(x) = x, S(y) = xy.$$

It is easy to verify that for every $\lambda \in \mathbb{C}$ the following expression defines an universal R-matrix of \mathcal{S}

$$\begin{aligned} \mathcal{R}_\lambda = & \frac{1}{2} (1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) + \\ & \frac{\lambda}{2} (y \otimes y + y \otimes xy + xy \otimes xy - xy \otimes y). \end{aligned}$$

It is also easy to verify that $\mathcal{R}_\lambda^{-1} = \tau_{\mathcal{S}, \mathcal{S}}(\mathcal{R}_\lambda)$. After a simple calculation we get $u = v = x$. This shows that $\theta = 1$ defines a ribbon element in \mathcal{S} . (Majid [9] Example 2.1.11 takes x as a ribbon element, but x is not central.) So for every λ we get a ribbon category with trivial twist.

Using theorem 2.12 we can show where to find the ribbon categories that we need in order to construct the so called quantum invariants. Let g be a semi-simple complex Lie algebra of finite dimension and $U(g)$ its universal enveloping algebra. Drinfeld [2, 3] defined a quantization of this algebra by introducing a formal parameter h in the defining relations between the generators of $U(g)$ that destroys its commutativity and cocommutativity. We denote this 'standard' quantization, which is defined over the ring $\mathbb{C}[[h]]$, by $U_h(g)$. Drinfeld also proved that $U_h(g)$ is a quasi-triangular Hopf algebra. Turaev and Reshetikhin [10] showed that $U_h(g)$ is actually a ribbon algebra by proving that the square root of $vu \in U_h(g)$ exists

and that it satisfies the properties (2.21). Theorem 2.12 shows that ${}_{U_h(g)}\mathcal{M}$ is a ribbon category, so by the universality property there exists for every module V in ${}_{U_h(g)}\mathcal{M}$ a unique functor $F_V : \mathcal{FT} \rightarrow {}_{U_h(g)}\mathcal{M}$ such that $F_V(+)=V$ and $F_V(-)=*V$. As the functor is defined on equivalence classes of framed tangles, we get an invariant of framed links when we restrict our functor to this particular kind of framed tangles. It can be shown that, if V is a finite dimensional g -module, there exists a unique $U_h(g)$ -module \tilde{V} of finite rank, defined over $\mathbb{C}[[h]]$, such that $\tilde{V} \equiv V \bmod h$. So, given a semi-simple finite dimensional complex Lie algebra g and a finite dimensional g -module V , we can define a unique framed knot and link invariant that essentially comes from the quantization of $U(g)$. That is why these invariants are called quantum invariants. Of course we could choose a right module W , instead of a left module. Then there is a unique functor $F_W : \mathcal{FT} \rightarrow \mathcal{M}_{U_h(g)}$ such that $F_W(+)=W$ and $F_W(-)=*W$.

There are now two questions that we want to discuss. The first is how to derive invariants of ordinary knots and links from this beautifully designed theory. The second is how do enhanced R-matrices fit into this picture. This last question is especially interesting because not all known R-matrices were found by evaluating the universal R-matrix of a quasi-triangular Hopf algebra (see for example [4], [7]).

The first problem can be solved easily. If you look closely at all arguments, you see that the whole theory would work for ordinary tangles as well, if only we required the twist on $+$ in \mathcal{T} to be trivial and the twist on a specified object in another ribbon category to be trivial as well.

Lemma 2.13. *The category of tangles \mathcal{T} is a ribbon category if we define the twist in the following way:*

$$\theta_+(\downarrow) = \downarrow \quad \text{and} \quad \theta_-(\uparrow) = \uparrow .$$

The definition of the twist is extended to all other tangles by applying formulas (2.16) and (2.17).

Proof. Trivial.

Theorem 2.14. *Let C be a ribbon category and V an object in C such that $\theta_V = \text{id}_V$. Then there exists a unique functor $F_V : \mathcal{T} \rightarrow C$, preserving braiding, duality and twist, that sends $+$ to V and $-$ to $*V$.*

Proof. The proof of this theorem is identical to that of theorem 2.9, apart from the details concerning the twist. Since we require F_V to preserve the twist, the twist in \mathcal{T} defined in the lemma above, is exactly the one that makes the construction of F_V possible. \blacksquare

So we have to know where to find this particular kind of ribbon categories. If H is a ribbon algebra with trivial ribbon element, then of course its category of representations ${}_H\mathcal{M}$ is such a category. Thus the condition $vu = 1$ seems a good one, if we already know that H is a quasi-triangular Hopf algebra. However there is one problem with this condition. It is not a condition on R -matrices but rather on quasi-triangular Hopf algebras, so enhancement does not seem to come into the picture. Nevertheless the next lemma shows that we are on the right way, by making a link to what one might call the 'dual' theory. This lemma follows more or less directly from [9] Prop. 4.2.2. Nevertheless, we have included a proof, because we could not find the lemma in the literature.

Lemma 2.15. *Let H be a quasi-triangular Hopf algebra with universal R -matrix \mathcal{R} . Let M be a finite dimensional H -module, so that $\rho_M(\mathcal{R}) = R$ is the R -matrix on $M \otimes M$. This R -matrix is biinvertible (see introduction). If we define*

$$\tilde{R} = \left((R^{t_2})^{-1} \right)^{t_2},$$

then we get $U_j^i = \rho_M(u)_j^i = \tilde{R}_{ja}^{ai}$ and $V_j^i = \rho_M(v)_j^i = \tilde{R}_{aj}^{ia}$.

Proof. We prove that R is biinvertible by showing that

$$\rho_M((\text{id} \otimes S) \mathcal{R}) = \tilde{R}.$$

Let $\mathcal{R} = \sum s_i \otimes t_i$. Then we get

$$\begin{aligned} 1 &= (\text{id} \otimes \varepsilon)(\mathcal{R}) = (\text{id} \otimes (S * \text{id}))(\mathcal{R}) = (\text{id} \otimes \mu)(\text{id} \otimes S \otimes \text{id})(\text{id} \otimes \Delta)(\mathcal{R}) \\ &= (\text{id} \otimes \mu)(\text{id} \otimes S \otimes \text{id})(\mathcal{R}_{13}\mathcal{R}_{12}) = \sum s_i s_j \otimes S(t_j) t_i. \end{aligned}$$

So we see

$$\begin{aligned} \delta_c^a \delta_d^b &= \rho_M(\sum s_i s_j \otimes S(t_j) t_i)_{cd}^{ab} = \sum \rho_M(s_i)_\alpha^a \rho_M(s_j)_c^\alpha \rho_M(S(t_j))_\beta^b \rho_M(t_i)_d^\beta \\ &= \rho_M(\sum s_i \otimes t_i)_{\alpha d}^{a\beta} \rho_M(\sum s_j \otimes S(t_j))_{c\beta}^{\alpha b} = R_{\alpha d}^{a\beta} \tilde{R}_{c\beta}^{\alpha b}. \end{aligned}$$

The proof of the other two assertions now follows easily:

$$U_b^a = \rho_M \left(\sum S(t_i) s_i \right)_b^a = \tilde{R}_{b\alpha}^{\alpha a},$$

$$V_b^a = \rho_M \left(\sum s_i S(t_i) \right)_b^a = \tilde{R}_{ab}^{a\alpha}. \blacksquare$$

The key observation here is that the matrices U and V can be derived directly from the R-matrix R , without using any information coming from H . So, if one is optimistic, one hopes that the condition $VU = I$, for any biinvertible R-matrix R , is the one equivalent to enhancement.

3. The dual theory

In the previous section we wrote that lemma 2.15 defined a link with the 'dual' theory. In this section we explain what we mean by that.

Definition 3.1. Let R be an $n^2 \times n^2$ R-matrix. If $K \langle T \rangle = K \langle T_1^1, \dots, T_n^n \rangle$ is the free bialgebra on n^2 generators, defined by

$$\Delta(T_j^i) = T_a^i \otimes T_j^a, \quad \varepsilon(T_j^i) = \delta_j^i,$$

then we define the bialgebra $A(R)$ to be

$$K \langle T \rangle / (RT_1T_2 - T_2T_1R).$$

For the proof that $J = (RT_1T_2 - T_2T_1R)$ is really a bialgebra ideal see for example [5]. This bialgebra is not quasi-triangular, but it has a 'dual' property.

Definition 3.2. Let H be a bialgebra or a Hopf algebra. We say that H is dual quasi-triangular if there exists a convolution-invertible map $R : H \otimes H \rightarrow \mathbb{C}$ such that

$$\sum b_{(1)} a_{(1)} R(a_{(2)} \otimes b_{(2)}) = \sum R(a_{(1)} \otimes b_{(1)}) a_{(2)} b_{(2)}, \quad (3.1)$$

$$\begin{aligned} R(ab \otimes c) &= \sum R(a \otimes c_{(1)}) R(b \otimes c_{(2)}), \\ R(a \otimes bc) &= \sum R(a_{(1)} \otimes c) R(a_{(2)} \otimes b). \end{aligned} \quad (3.2)$$

for all $a, b, c \in H$.

Here we used Sweedler's notation: $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$. One can check that $A(R)$ is a dual quasi-triangular bialgebra with

$$R(T_c^a \otimes T_d^b) = R_{cd}^{ab}.$$

In a way $A(R)$ is the universal bialgebra with dual quasi-triangular structure defined by R .

Theorem 3.3. *Let H be a dual quasi-triangular bialgebra with n^2 generators $\{x_1^1, \dots, x_n^n\}$ and 1, having the coalgebra structure given by*

$$\Delta(x_j^i) = x_a^i \otimes x_j^a, \quad \varepsilon(x_j^i) = \delta_j^i.$$

If

$$R(x_a^i \otimes x_b^j) = R_{ab}^{ij},$$

then R is an R -matrix and H is a quotient of $A(R)$.

In general $A(R)$ does not define a Hopf algebra. Even after dividing out by some extra relations we do not always get a Hopf algebra. However, if R is biinvertible, then Majid [9] showed that there is a formal way to extend $A(R)$ to a Hopf algebra $H(R)$.

Theorem 3.4. *a) Suppose that we can add relations to $A(R)$ such that the dual quasitriangular structure descends to the quotient and gives us a dual quasi-triangular Hopf algebra. Then*

$$R(T \otimes T^{-1}) = \tilde{R}$$

obeys

$$\tilde{R}_{aj}^{ib} R_{lb}^{ak} = \delta_l^i \delta_j^k = R_{aj}^{ib} \tilde{R}_{lb}^{ak}, \quad \text{i.e. } \tilde{R} = \left((R^{t_2})^{-1} \right)^{t_2}.$$

Moreover,

$$S^2 T = V^{-1} T V = U T U^{-1}; \quad V_j^i = \tilde{R}_{aj}^{ia}, \quad U_j^i = \tilde{R}_{ja}^{ai},$$

$$V_2 = R V_2 \tilde{R}, \quad V_1 = \tilde{R} V_1 R.$$

b) If R is biinvertible, then we can enlarge $A(R)$ to obtain a Hopf algebra $H(R)$, with the same dual quasi-triangular structure, by adding formally the generators $T^{-1} = \left((T^{-1})_j^i \right)_{i,j=1}^n$, with coalgebra structure

$$\Delta(T^{-1}) = (T^{-1} \otimes T^{-1}), \quad \varepsilon(T^{-1}) = I,$$

and additional relations

$$TT^{-1} = T^{-1}T = I,$$

$$RT_1 = T_2T_1RT_2^{-1}, \quad T_2^{-1}RT_1T_2 = T_1R, \quad T_1^{-1}T_2^{-1}R = RT_2^{-1}T_1^{-1}.$$

The antipode we define by

$$ST = T^{-1}, \quad ST^{-1} = V^{-1}TV = UTU^{-1},$$

and the dual quasi-triangular structure by

$$R(T \otimes T) = R(T^{-1} \otimes T^{-1}) = R, \quad R(T^{-1} \otimes T) = R^{-1}, \quad R(T \otimes T^{-1}) = \tilde{R}.$$

The proof of part a) in the previous theorem can be found in [9] Prop. 4.2.2. Part b) follows more or less automatically from Majid's comments after this proposition. The universality of $A(R)$ and the theorem above show us the following.

Corollary 3.5. *Let H be a dual quasi-triangular Hopf algebra with $2n^2$ generators $\{x_1^1, \dots, x_n^n, y_1^1, \dots, y_n^n\}$ and 1, satisfying the relations*

$$XY = I, \quad X_j^i = x_j^i, \quad Y_j^i = y_j^i.$$

Suppose that H has the coalgebra structure given by

$$\Delta(X) = X \otimes X, \quad \Delta(Y) = Y \otimes Y, \quad \varepsilon(X) = \varepsilon(Y) = I.$$

If

$$R(X \otimes X) = R,$$

then R is biinvertible and H is a quotient of $H(R)$. Furthermore

$$R(Y \otimes X) = R^{-1}, \quad R(X \otimes Y) = \tilde{R}, \quad R(Y \otimes Y) = R,$$

$$S(X) = Y, \quad S(Y) = V^{-1}XV = UXU^{-1}.$$

In this context theorem 2.3 becomes

Theorem 3.6. *Let H be a Hopf algebra. If H is dual quasi-triangular, then the category of finite dimensional left H -comodules ${}^H\mathcal{M}$ is a braided category. Conversely, if H is finite dimensional and ${}^H\mathcal{M}$ is a braided category, then H is a dual quasi-triangular Hopf algebra.*

Sketch of a proof. If H is dual quasi-triangular, then ${}^H\mathcal{M}$ becomes a braided category with braiding

$$c_{V,W}(v \otimes w) = \sum R(v^{(1)} \otimes w^{(1)}) w^{(2)} \otimes v^{(2)},$$

for all H -modules V and W , where we define the comodule structure explicitly by $\Delta_V(v) = \sum v^{(1)} \otimes v^{(2)}$. If H is finite dimensional and ${}^H\mathcal{M}$ is a braided category, then we define the dual quasi-triangular structure on H by

$$R(g \otimes h) = (\varepsilon \otimes \varepsilon) \tau_{H,H} c_{H,H}(g \otimes h). \blacksquare$$

Of course there is a similar theorem for the category of right comodules \mathcal{M}^H . In that case the braiding is defined by

$$c_{V,W}(v \otimes w) = \sum w^{(1)} \otimes v^{(1)} R(v^{(2)} \otimes w^{(2)}).$$

We see that, if R is biinvertible, the category ${}^{H(R)}\mathcal{M}$ is a braided category. If V is the left fundamental corepresentation V , with coaction $e^j \mapsto T_a^j \otimes e^a$, then the braiding is defined by

$$c_{V,V}(e^i \otimes e^j) = R_{ab}^{ij} e^b \otimes e^a. \quad (3.3)$$

In the same way the category $\mathcal{M}^{H(R)}$ is braided. If W is the right fundamental corepresentation W , with coaction $e_j \mapsto e_a \otimes T_j^a$, then the braiding is given by

$$c_{W,W}(e_i \otimes e_j) = e_b \otimes e_a R_{ij}^{ab}. \quad (3.4)$$

If $*V$ is the dual of the left fundamental corepresentation with dual basis $\{f_i\}$, then the coaction on $*V$ is given by $f_i \mapsto S(T_i^a) \otimes f_a$. The braiding on tensor products of V and $*V$ is now given by (3.3) and the following formulas

$$c_{*V,*V}(f_i \otimes f_j) = R(ST_i^a \otimes ST_j^b) f_b \otimes f_a = R_{ij}^{ab} f_b \otimes f_a, \quad (3.5)$$

$$c_{V,*V}(e^i \otimes f_j) = R(T_a^i \otimes ST_j^b) f_b \otimes e^a = \tilde{R}_{aj}^{ib} f_b \otimes e^a, \quad (3.6)$$

$$c_{*V,V}(f_i \otimes e^j) = R(ST_i^a \otimes T_b^j) e^b \otimes f_a = (R^{-1})_{ib}^{aj} e^b \otimes f_a. \quad (3.7)$$

Let $*W$ be the dual of the right fundamental corepresentation, with dual basis $\{f^i\}$ and coaction $f^i \mapsto f^a \otimes S(T_a^i)$. Then the braiding is defined by (3.4) and

$$c_{*W,*W}(f^i \otimes f^j) = R(ST_a^i \otimes ST_b^j) f^b \otimes f^a = f^b \otimes f^a R_{ab}^{ij},$$

$$c_{W,*W}(e_i \otimes f^j) = R(T_i^a \otimes ST_b^j) f^b \otimes e_a = f^b \otimes e_a \tilde{R}_{ib}^{aj},$$

$$c_{*W,W}(f^i \otimes e_j) = R(ST_b^i \otimes T_j^a) e_a \otimes f^b = e_a \otimes f^b (R^{-1})_{bj}^{ia}.$$

So ${}^{H(R)}\mathcal{M}$ and $\mathcal{M}^{H(R)}$ are braided categories with left duality. The next definition dualizes the notion of ribbon algebra.

Definition 3.7. Let H be a dual quasi-triangular Hopf algebra with $2n^2$ generators $\{x_1^1, \dots, x_n^n, y_1^1, \dots, y_n^n\}$ and 1, such that it satisfies the hypotheses in corollary 3.5. Then we call H a coribbon algebra if VU is invertible and has a central square root Θ , which we call the coribbon element on H , such that it defines an element of H^* by putting

$$\Theta(x_j^i) = \Theta_j^i, \quad \Theta(y_j^i) = (\Theta^{-1})_j^i, \quad \Delta(\Theta) = (R_{21}R)^{-1}(\Theta \otimes \Theta),$$

$$\varepsilon(\Theta) = 1, \quad S(\Theta) = \Theta.$$

Here we mean by the Hopf algebra structure the one in H^* .

This gives us an analogue of theorem 2.12.

Theorem 3.8. Let H satisfy the previous hypotheses. Then H is a coribbon algebra iff ${}^H\mathcal{M}$ is a ribbon category (iff \mathcal{M}^H is a ribbon category).

Proof. The proof is just the dual version of the proof of theorem 2.12. ■

The next theorem dualizes theorem 2.9.

Theorem 3.9. *let R be a biinvertible R -matrix. If there is a coribbon element Θ on $H(R)$, then R and Θ define a unique functor $F_{RP} : \mathcal{FT} \rightarrow^{H(R)} \mathcal{M}$ that preserves braiding, duality and twist such that $F_{RP}(+) = V$ and $F_{RP}(-) = {}^*V$ with V the left fundamental corepresentation of $H(R)$. They also define a unique functor $F_{PR} : \mathcal{FT} \rightarrow \mathcal{M}^{H(R)}$ that preserves braiding, duality and twist such that $F_{PR}(+) = W$ and $F_{PR}(-) = W^*$ with W the right fundamental corepresentation of $H(R)$.*

Proof. Just dualize the proof of theorem 2.9. ■

Note that we have to specify the coribbon element because there might be more than one.

Corollary 3.10. *Let R be a biinvertible R -matrix. If $\Theta = I$ defines a coribbon element on $H(R)$, then there exists a unique functor $F_{RP} : \mathcal{T} \rightarrow^{H(R)} \mathcal{M}$ that preserves braiding, duality and twist such that $F_{RP}(+) = V$ and $F_{RP}(-) = {}^*V$ with V the left fundamental corepresentation of $H(R)$. Furthermore there exists a unique functor $F_{PR} : \mathcal{FT} \rightarrow \mathcal{M}^{H(R)}$ that preserves braiding, duality and twist such that $F_{PR}(+) = W$ and $F_{PR}(-) = {}^*W$ with W the right fundamental corepresentation of $H(R)$.*

Proof. Just dualize the proof of corollary 2.14. ■

4. Enhancement

In this section we show exactly how enhancement fits into the setup of the previous section. We first recall Turaev's theorem [12].

Theorem 4.1. *Given an enhanced R -matrix in the sense of definition 1.2, there exists a unique tensor functor $F : \mathcal{T} \rightarrow \mathcal{V}_\mathbb{C}$ such that $F(+) = V, F(-) = V^*$, and*

$$\begin{aligned} F(X^+) &= c, & F(\cup) &= \text{coev}_V, & F(\cup^-) &= (\text{id}_{V^*} \otimes \mu^{-1}) \text{coev}_{V^*}, \\ F(X^-) &= c^{-1}, & F(\cap) &= \text{ev}_V, & F(\cap^-) &= \text{ev}_{V^*}(\mu \otimes \text{id}_{V^*}). \end{aligned}$$

Conversely, let $F : \mathcal{T} \rightarrow \mathcal{V}_\mathbb{C}$ be a representation of \mathcal{T} such that $F(+) = V, F(-) = V^$ and*

$$\begin{aligned} F(X^+) &= c, & F(\cup) &= \text{coev}_V, & F(\cup^-) &= b'_V, \\ F(X^-) &= c^{-1}, & F(\cap) &= \text{ev}_V, & F(\cap^-) &= d'_V, \end{aligned}$$

where c is an automorphism of $V \otimes V$. Then there exists a unique automorphism μ of V such that (c, μ) is an enhanced R -matrix in the sense of def.1.2 and

$$b'_V = (\text{id}_{V^*} \otimes \mu^{-1}) \text{coev}_{V^*}, \quad d'_V = \text{ev}_{V^*} (\mu \otimes \text{id}_{V^*}).$$

Sketch of a proof. We only sketch the second part of the theorem, because we need it for the proof of our own results. For the rest of the proof we refer to [12] (see also [8]). Let $\{v_1, \dots, v_n\}$ be a basis of V and $\{w_1, \dots, w_n\}$ the dual basis of V^* . We define

$$b'_V(1) = \sum_{i,j} B_{i,j} v_i \otimes w_j, \quad d'_V(v_i \otimes w_j) = D_{i,j}.$$

Since F is a representation of the tangle category we get

$$(d'_V \otimes \text{id}_V) (\text{id}_V \otimes b'_V) = \text{id}_V, \quad (\text{id}_{V^*} \otimes d'_V) (b'_V \otimes \text{id}_{V^*}) = \text{id}_{V^*}.$$

So $BD = DB = I$. This allows us to define an automorphism $\beta : V^* \rightarrow V^*$ by

$$\beta(w_j) = \sum_i B_{i,j} w_i.$$

Take $\mu = (\beta^{-1})^t$. Then reading Turaev's proof shows that (c, μ) is the enhanced R -matrix with the desired properties. The automorphism μ is unique, because it is completely determined by the requirement $d'_V = \text{ev}_{V^*} (\mu \otimes \text{id}_{V^*})$. ■

Next we give our results of this article.

Theorem 4.2. *Let R be a biinvertible R -matrix. If there exists a complex number $\alpha \in \mathbb{C}^*$ such that $VU = \alpha^2 I$, then $\Theta = \alpha I$ is a coribbon element on $H(R)$. So R and Θ define a unique functor $F_{RP} : \mathcal{FT} \rightarrow^{H(R)} \mathcal{M}$ that preserves braiding, duality and twist such that $F_{RP}(+) = V$ and $F_{RP}(-) = {}^*V$ with V the left fundamental corepresentation of $H(R)$, and there is a unique functor $F_{PR} : \mathcal{FT} \rightarrow \mathcal{M}^{H(R)}$ that preserves braiding, duality and twist such that $F_{PR}(+) = W$ and $F_{PR}(-) = {}^*W$ with W the right fundamental corepresentation of $H(R)$.*

Proof. Obviously $\Theta = \alpha I$ defines a coribbon element on $H(R)$ if we can prove that Θ is well defined in H^* . We only prove $\Theta(TT^{-1}) = I$, since the other identities follow from the fact that Θ is central.

$$\Theta(T_\gamma^i (T^{-1})_l^\gamma) = \left(\tilde{R}^{-1}\right)_{\alpha\beta}^{i\gamma} \left(\tilde{R}^{-1}\right)_{ba}^{\beta\alpha} \Theta_\gamma^a \Theta_l^b = \alpha^2 \left(\tilde{R}^{-1}\right)_{\alpha\beta}^{i\gamma} \left(\tilde{R}\right)_{l\gamma}^{\beta\alpha}.$$

Since we have

$$\left(\tilde{R}^{-1}\right)_{\alpha\beta}^{i\gamma} \left(\tilde{R}^{-1}\right)_{l\gamma}^{\beta\alpha} \left(\tilde{R}\right)_{\gamma\beta}^{l\gamma} \left(\tilde{R}\right)_{l\alpha}^{\alpha\beta} = \delta_l^i$$

and

$$\left(\tilde{R}\right)_{\gamma\beta}^{l\gamma} \left(\tilde{R}\right)_{l\alpha}^{\alpha\beta} = V_\beta^l U_l^\beta = (VU)_l^l = \alpha^2,$$

we get

$$\Theta \left(T_\gamma^i (T^{-1})_\gamma^l\right) = \delta_l^i.$$

So Θ defines a coribbon element on $H(R)$. The rest follows from theorem 3.9. ■

Corollary 4.3. *Suppose $\alpha = 1$. Then there is a unique functor $F_{RP} : \mathcal{T} \rightarrow {}^{H(R)}\mathcal{M}$ preserving braiding, duality and twist such that $F_{RP}(+) = V$ and $F_{RP}(-) = {}^*V$ and a unique functor $F_{PR} : \mathcal{T} \rightarrow \mathcal{M}^{H(R)}$ that preserves braiding, duality and twist such that $F_{PR}(+) = W$ and $F_{PR}(-) = {}^*W$.*

Proof. This follows from corollary 3.10. ■

Corollary 4.4. *If $VU = \alpha^2 I$, then the matrix $R' = \alpha R$ satisfies the condition in the previous corollary.*

Proof. Trivial.

Theorem 4.5. *If R is a biinvertible R -matrix and $VU = I$, then (PR, U) and (RP, V) are enhanced R -matrices in the sense of definition 1.2.*

Proof. The functors F_{PR} and F_{RP} in corollary 4.3 satisfy the hypotheses in theorem 4.1. In the first case we get $\mu = (\beta^{-1})^t = U$ and in the second case $\mu = (\beta^{-1})^t = V$. ■

Corollary 4.6. *If R is a biinvertible R -matrix and $VU = \alpha^2 I$, then $(\alpha PR, \alpha^{-1}U)$ and $(\alpha RP, \alpha^{-1}V)$ are enhanced R -matrices in the sense of def.1.2. Of course $(PR, U, \alpha^{-1}, \alpha)$ and $(RP, V, \alpha^{-1}, \alpha)$ are enhanced R -matrices in the sense of def.1.1.*

Proof. Obvious. ■

Our next lemma shows that the condition of R being biinvertible is no restriction at all, when one considers enhancement in the sense of def.1.2.

Lemma 4.7. *Let R be an invertible R -matrix. If R can be enhanced in the sense of def.1.2, then R is biinvertible.*

Proof. Suppose that (PR, μ) is enhanced in the sense of def.1.2. One of the relations satisfied by the basic elements of \mathcal{T} (see [8]) is the following

$$Y^- \circ T^+ = \text{id}_{V^*} \otimes \text{id}_V.$$

Note that this relation corresponds to an extended version of the second Reidemeister move. When we apply the functor F to the LHS of this relation and use lemma 2.10 we find

$$F(Y^- \circ T^+)(f_a \otimes e^b) = R_{\beta a}^{b\alpha} (\mu^{-1})_{\delta}^{\gamma} (R^{-1})_{\varepsilon \theta}^{\delta \beta} \mu_{\alpha}^{\varepsilon} f_{\gamma} \otimes e^{\theta}.$$

If we take

$$A_{\theta \alpha}^{\beta \gamma} = (\mu^{-1})_{\delta}^{\gamma} (R^{-1})_{\varepsilon \theta}^{\delta \beta} \mu_{\alpha}^{\varepsilon},$$

then we get

$$R_{\beta a}^{b\alpha} A_{\theta \alpha}^{\beta \gamma} = \delta_{\gamma}^a \delta_{\theta}^b.$$

Thus we see that R is biinvertible with

$$A = \left((R^{t_2})^{-1} \right)^{t_2} = \tilde{R}.$$

There is a similar proof if (RP, ν) is enhanced. ■

Note that our proof also implies that the functor F is actually a functor to the category of $H(R)$ -comodules satisfying

$$F(Y^-) = c_{V, V^*}, \quad F(T^+) = c_{V^*, V}^{-1}.$$

The same reasoning applied to the identity

$$Y^+ \circ T^- = \text{id}_{V^*} \otimes \text{id}_V$$

shows

$$F(Y^+) = c_{V,V^*}^{-1}, \quad F(T^-) = c_{V^*,V}.$$

Finally we can derive directly

$$F(Z^\pm) = c_{V^*,V^*}^\pm$$

by using lemma 2.10. Next we show that our condition is a necessary condition for enhancement.

Theorem 4.8. *If (PR, μ) is an enhanced R -matrix in the sense of def.1.2, then R is biinvertible and $VU = I$. If (RP, ν) is an enhanced R -matrix in the sense of def.1.2, then R is biinvertible and $VU = I$.*

Proof. Let us show the first case, the second being proven in a similar way. If we take V to be the fundamental left $H(R)$ -comodule, then theorem 4.1 shows that (PR, μ) defines a functor F from \mathcal{T} to $\mathcal{V}_{\mathbb{C}}$ with $F(+)=V$ and $F(-)=V^*$. We define the subcategory $C(V)$ of $\mathcal{V}_{\mathbb{C}}$ generated by all tensor powers of V and V^* . This subcategory becomes a ribbon category if we define the braiding by the formulas (3.3), (3.5), (3.6), (3.7) and the left duality by the usual evaluation and coevaluation maps. The twist is defined by $F(\varphi) = \theta_V : V \rightarrow V$. We define θ_{*V} and $\theta_{V \otimes V}, \theta_{V \otimes V^*}, \theta_{V^* \otimes V}, \theta_{V^* \otimes V^*}$ by the formulas 2.16 and 2.17. Note that $C(V)$ is not a ribbon subcategory of $\mathcal{V}_{\mathbb{C}}$. For example $V^{**} \neq V$ as $H(R)$ -comodules. The previous lemma and the comments thereafter show that R is biinvertible and the functor F maps \mathcal{T} onto this subcategory, preserving braiding, duality and twist. By lemma 2.8 we know how θ_V^{-2} acts. Using the second expression in (2.20) we get

$$\begin{aligned} \theta_V^{-2}(e^i) &= (\text{ev}_V c_{V,V^*} \otimes \text{id}_V) (\text{id}_V \otimes c_{V,V^*} \text{coev}_V) (e^i) \\ &= R \left(T_\alpha^i \otimes ST_b^\beta \right) R \left(T_a^j \otimes ST_j^b \right) \langle f_\beta, e^\alpha \rangle e^a = V_b^i U_a^b e^a = (VU)_a^i e^a. \end{aligned}$$

So $\theta_V^{-2} = VU$. Since Reidemeister move 1 is available in \mathcal{T} , we must have $\theta_V = I$. Thus we get

$$VU = I.$$

By using the right fundamental $H(R)$ -comodule and applying the same arguments we get the result for the matrix RP . ■

Our last theorem proves that if a biinvertible R -matrix can be enhanced, there is only one way to do it.

Theorem 4.9. *If (PR, μ) is an enhanced R -matrix in the sense of def. 1.2, then R is biinvertible and $\mu = U$. If (RP, ν) is an enhanced R -matrix in the sense of def. 1.2, then R is biinvertible and $\nu = V$.*

Proof. Let V be the left fundamental $H(R)$ -comodule. If (PR, μ) is an enhanced R -matrix, then, as we showed in the previous theorem, R is biinvertible and there is a unique functor $F : \mathcal{T} \rightarrow C(V)$ that preserves braiding, left duality and twist. Of course F also preserves right duality, so

$$F(\cup^-) = (\text{id}_{V^*} \otimes \mu^{-1}) \text{coev}_{V^*}, \quad F(\cap^-) = \text{ev}_{V^*}(\mu \otimes \text{id}_{V^*}) \quad (4.1)$$

define a right duality in $C(V)$. On the other hand we know that

$$Y^- \circ \cup = \cup^-, \quad \cap \circ Y^- = \cap^-$$

are satisfied in \mathcal{T} . Hence

$$F(\cup^-) = F(Y^-) F(\cup) = c_{V,V^*} \text{coev}_V, \quad F(\cap^-) = F(\cap) F(Y^-) = \text{ev}_V c_{V,V^*}.$$

So both right dualities are the same;

$$\begin{aligned} (\text{id}_{V^*} \otimes \mu^{-1}) \text{coev}_{V^*} &= c_{V,V^*} \text{coev}_V, \\ \text{ev}_{V^*}(\mu \otimes \text{id}_{V^*}) &= \text{ev}_V c_{V,V^*}. \end{aligned}$$

As a result we get

$$\mu_j^i = \text{ev}_{V^*}(\mu \otimes \text{id}_{V^*})(e^i \otimes f_j) = \text{ev}_V c_{V,V^*}(e^i \otimes f_j) = \tilde{R}_{aj}^{ib} \langle f_b, e^a \rangle = U_j^i.$$

In the same way one can prove the statement for the matrix RP . ■

Note that theorem 4.8 and 4.9 are only true for enhancement in the sense of def.1.2. If R is biinvertible and $(PR, U, 1, 1)$ is enhanced in the sense of def.1.1, then $UV = \text{Tr}_2((PR)^{-1} U_2) = I$ (see Appendix). If R is biinvertible and $(RP, V, 1, 1)$ is enhanced in the sense of def.1.1, then $VU = \text{Tr}_2((RP)^{-1} V_2) = I$ (see Appendix). But we don't know if there are no other μ and ν such that $(PR, \mu, 1, 1)$ and $(RP, \nu, 1, 1)$ are enhanced.

5. Examples

As an example we enhance all biinvertible R-matrices of dimension 2.

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tilde{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^{-1} & 0 & 0 \\ 0 & 0 & s^{-1} & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix}, V = \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

For $q = 1$ the matrices (PR, U) and (RP, V) are enhanced.

$$R = \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \end{pmatrix}$$

,

$$U = V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrices (PR, U) and (RP, V) are enhanced.

$$R = \begin{pmatrix} 1 & 1 & p & q \\ 0 & 1 & 0 & p \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 1 & -1 & -p & 2p - q \\ 0 & 1 & 0 & -p \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U = V = \begin{pmatrix} 1 & -p - 1 \\ 0 & 1 \end{pmatrix}.$$

For $p = -1$ the matrices (PR, U) and (RP, V) are enhanced.

$$R = \begin{pmatrix} 1 & 1 & -1 & q \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & -q \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 1 & -1 & 1 & -q^2 - q - 1 \\ 0 & 1 & 0 & -q \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 1+q \\ 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & -1-q \\ 0 & 1 \end{pmatrix}.$$

(PR, U) and (RP, V) are enhanced.

$$R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

R is not biinvertible.

$$R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U = V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(PR, U) and (RP, V) are enhanced.

$$R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tilde{R} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U = V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(PR, U) and (RP, V) are enhanced.

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & p & q - q^{-1} & 0 \\ 0 & 0 & p^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tilde{R} = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & p^{-1} & \frac{q^{-1}-q}{q^2} & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix},$$

$$U = \begin{pmatrix} q^{-1} & 0 \\ 0 & q^{-3} \end{pmatrix}, V = \begin{pmatrix} q^{-3} & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

$(q^{-2}PR, q^2U)$ and $(q^{-2}RP, q^2V)$ are enhanced.

$$R = \begin{pmatrix} q & 0 & 0 & q \\ 0 & p & q - q^{-1} & 0 \\ 0 & 0 & p^{-1} & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}, \tilde{R} = \begin{pmatrix} q^{-1} & 0 & 0 & -q \\ 0 & p^{-1} & q - q^{-1} & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & -q \end{pmatrix}$$

$$U = \begin{pmatrix} q^{-1} & 0 \\ 0 & -q^{-1} \end{pmatrix}, V = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}.$$

(PR, U) and (RP, V) are enhanced.

$$R = \begin{pmatrix} q - q^{-1} + 2 & 0 & 0 & q - q^{-1} \\ 0 & q + q^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & q + q^{-1} & 0 \\ q - q^{-1} & 0 & 0 & q - q^{-1} - 2 \end{pmatrix},$$

$$\tilde{R} = \frac{1}{4} \begin{pmatrix} q - q^{-1} + 2 & 0 & 0 & q^{-1} - q \\ 0 & q + q^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & q + q^{-1} & 0 \\ q^{-1} - q & 0 & 0 & q^{-1} - q - 2 \end{pmatrix},$$

$$U = V = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$(\frac{1}{2}PR, 2U)$ and $(\frac{1}{2}RP, 2V)$ are enhanced.

The permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is not biinvertible, as one can check easily. But, as we wrote in the introduction already, $(P^2, I, \frac{1}{2}, 2)$ is an enhanced matrix.

Appendix

In this appendix we give an elementary proof of corollary 4.5.

Theorem 4.10. *Let R be a biinvertible R -matrix. If $VU = UV = I$, then (RP, V) and (PR, U) are enhanced R -matrices in the sense of def.1.2.*

Proof. In order to prove that (RP, V) is an enhanced R -matrix we use the identities $V_1 = \tilde{R}V_1R$ and $V_2 = RV_2\tilde{R}$ (see [9]). We first prove that $V \otimes V = V_1V_2 = V_2V_1$ commutes with R :

$$RPV_1V_2 = RV_2PV_2 = RV_2V_1P = V_2\tilde{R}_1^{-1}VP = V_2V_1RP.$$

Next we prove the conditions on the partial traces:

$$\begin{aligned} (\text{Tr}_2(RPV_2))_c^a &= (RPV_2)_{cd}^{ad} = R_{ji}^{ad} (\text{id} \otimes V)_{cd}^{ij} = R_{ji}^{ad} \delta_c^i V_d^j = R_{jc}^{ad} \tilde{R}_{kd}^{jk} = \delta_k^a \delta_c^k = \delta_c^a, \\ (\text{Tr}_2((RP)^{-1}V_2))_c^a &= ((RP)^{-1}V_2)_{cd}^{ad} = (PR^{-1}V_2)_{cd}^{ad} = (PR^{-1}RV_2\tilde{R})_{cd}^{ad} \\ &= (PV_2\tilde{R})_{cd}^{ad} = (\text{id} \otimes V)_{ij}^{da} \tilde{R}_{cd}^{ij} = \delta_i^d V_j^a \tilde{R}_{cd}^{ij} = \tilde{R}_{kj}^{ak} \tilde{R}_{cd}^{dj} = V_j^a U_c^j = \delta_c^a. \end{aligned}$$

Finally we prove condition(1.11). It's not difficult (see [12] or [8]) to show that this condition is equivalent to the following condition

$$(\text{Id}_V \otimes (\mu^*)^{-1}) (S^{\pm 1}P)^{t_2} (\text{id}_V \otimes \mu^*) (PS^{\pm 1})^{t_2} = \text{id}_{V \otimes^* V}. \quad (4.2)$$

We prove that (RP, V) satisfies (4.2).

$$\begin{aligned} (V_2^{t_2})^{-1} R^{t_2} V_2^{t_2} (R^{-1})^{t_2} &= (V_2^{t_2})^{-1} R^{t_2} (R^{-1}V_2)^{t_2} = \\ (V_2^{t_2})^{-1} R^{t_2} (V_2\tilde{R})^{t_2} &= (V_2^{t_2})^{-1} R^{t_2} (R^{t_2})^{-1} V_2^{t_2} = I \end{aligned}$$

This proves the first identity in (4.2). The second follows from

$$\begin{aligned} (PR^{-1}P)^{t_2} V_2^{t_2} &= (V_2PR^{-1}P)^{t_2} = (PV_1R^{-1}P)^{t_2} = (P\tilde{R}V_1P)^{t_2} \\ &= (P\tilde{R}PV_2)^{t_2} = V_2^{t_2} (P\tilde{R}P)^{t_2} = V_2^{t_2} ((PRP)^{t_2})^{-1}. \end{aligned}$$

The other pair is proven to be an enhanced R -matrix in the same way, using the identities $U_1 = RU_1\tilde{R}$ and $U_2 = \tilde{R}U_2R$.

$$PRU_1U_2 = PU_1\tilde{R}_2^{-1}U_2 = PU_1U_2R = U_2PU_2R = U_2U_1PR$$

$$\begin{aligned} ((\text{Tr}_2(PRU_2))^t)^a_c &= \text{Tr}_2((PRU_2)^t)^a_c = \text{Tr}_2(U_2^t(PR)^t)^a_c = (\text{id} \otimes U)_{ad}^{ij} (PR)_{ij}^{cd} \\ &= \delta_a^i \tilde{R}_{dk}^{kj} R_{ij}^{dc} = \tilde{R}_{dk}^{kj} R_{aj}^{dc} = \delta_a^k \delta_k^c = \delta_a^c \end{aligned}$$

$$\begin{aligned} (\text{Tr}_2((PR)_2^{-1} U_2))^a_c &= ((PR)_2^{-1} U_2)_{cd}^{ad} = ((R^{-1} P U_2))_{cd}^{ad} = (R^{-1} U_1 P)_{cd}^{ad} \\ &= (U_1 \tilde{R} P)_{cd}^{ad} = (U \otimes \text{id})_{ij}^{ad} \tilde{R}_{dc}^{ij} = U_i^a \delta_j^d \tilde{R}_{dc}^{ij} = U_i^a V_c^i = \delta_c^a. \end{aligned}$$

The proof that (PR, U) satisfies (4.2) follows in an analogous way from the identities $U_1 = RU_1\tilde{R}$ and $U_2 = \tilde{R}U_2R$. ■

Corollary 4.11. *Let R be a biinvertible R -matrix. If there exists a scalar $\alpha \in \mathbb{C}^*$ such that*

$$\alpha^2 UV = I,$$

then $(\alpha PR, \alpha^{-1}U)$ and $(\alpha RP, \alpha^{-1}V)$ are enhanced R -matrices in the sense of def.1.2 and $(PR, U, \alpha^{-1}, \alpha)$ and $(RP, V, \alpha^{-1}, \alpha)$ are enhanced in the sense of def.1.1.

Proof. This clearly follows from the theorem above. ■

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